International Journal of Theoretical Physics, Vol. 20, No. 10, 1981

Space-Time Trajectories of Quantum Particles

Daniel C. Galehouse[1]

Department of Physics, University of California, Berkeley, California

Received May 13, 1980

The concept of trajectory is extended theoretically from classical mechanics through nonrelativistic and relativistic quantum mechanics. Forced motion of the particle as might be caused by an electromagnetic field is included in the equations. A new interpretation of the electromagnetic potential and the gauge transformation is presented. Using this formal structure, the problem of collecting particles into packets using trajectories is studied for both quantum mechanics and classical mechanics. Quantum mechanical trajectories are found to be significantly more restricted than those allowed by classical physics. The uncertainty principle comes from the second-order nature of the field equation without recourse to statistical arguments. The trajectories of particles in a quantum state can be calculated explicitly from the wave function (also see article in Volume 20, Number 6).

1. INTRODUCTION

Mechanics is the description of the motion caused by forces acting on matter. Any fundamental set of rules of mechanics attempts to divide matter into the smallest possible identifiable units. These units, called particles, may have few identifying features other than a particular position in space at a particular time. It is deduced that the macroscopic forces which cause the motion of large particles arise from intrinsic forces of the smallest possible individual particles. Since Newton, mechanics has successfully described the motion of large macroscopic particles in conjunction with overall average forces.

These classical laws of Newton, derived empirically without knowledge of any fundamental laws of motion, are known to fail if applied directly to very small systems. Quantum mechanics must be applied in these cases and is verified by experiment. During the process of adapting classical mecha-

[1]Presently at General Electric Corp., Edison Park, 8499 Darrow Road, Twinsburg, Ohio 44087.

0020-7748/81/1000-0787$03.00/0

nics laws to quantum phenomena, the concept of a particle having an identity depending upon a position in space and time has not been retained, and other functional mathematical structures have been substituted. Thus, it is that the possibility of defining the motion of a point particle for quantum mechanics has been discarded from accepted physical dogma earlier in this century. Nevertheless, the functional mathematical structures have been sufficiently successful that there is no technological need for a quantum point particle.

This article reevaluates this problem from a point of view not before discussed adequately; namely, the fundamental mechanics of quantum point particles differs only in minor details from the mechanics of classical particles. It is expected that explanation of this point of view will set the context for further discussion of geometrical theories of quantum mechanics in a succeeding article (Galehouse, 1981). Eventually, it may be possible to connect the principally statistical formulation of quantum based physics with the rigidly geometrical theories of gravitation.

2. BASIC STRUCTURE OF PHYSICAL LAWS, DYNAMICS, AND KINEMATICS

One of the basic conceptual separations of the laws of mechanics can be found in the work of Issac Newton. (Newton, 1934, p. 13) The important part of his law of motion expressed in modern form

$$F=ma \tag{1}$$

contains two elements: dynamics, expressed more or less in the symbol F for force, and kinematics, contained more or less in the acceleration a. The mass, m, is an essential constant connecting the two different types of mathematical quantities. Certain other properties of this equation are important: (1) It is of second order in one variable, the time; and (2) it is linear–two forces produce an acceleration which is the sum of the accelerations produced by each force alone. The forces may be externally imposed or may be inferred from the equation itself. This equation was developed for macroscopic particles, viz., cannon balls, and the nature of properties of the equation are the ones most appropriate for this type of problem.

In the microscopic world, none of these properties is necessarily useful; First-order or even algebraic equations may be more useful. The forces are so strongly dependent on position that linearity of the time derivative and linearity of force addition are not useful. The mass is not continuously adjustable (such as the cannon ball) but is a constant, at least for stable particles, and has only certain discrete values. Microscopic electromagnetic interactions are treated in terms of the vector potential without reference to

forces. The esoteric forces are replaced by other esoteric things, and because the microscopic world is so far beyond the direct reach of the senses, a physical language has been developed which corresponds to and explains this mathematical structure. Lastly, even the distinction between kinematics and dynamics may disappear for a microscopic mechanics. This separation is completely dependent upon the availability of equivalent particles differing only in mass. By experiment, the smallest particles have not been found to have a continuous distribution of allowed masses. Mass is discrete, much like energy in bound quantum systems; and no meaningful experiment to separate kinematics from dynamics can be performed.

Furthermore, if the theory is to be geometrical, the mass must be a geometrical constant. The kinematics can only be geometrical, and so then must be the dynamics. Such a description, which describes the motion of particles as trajectories has not the need for both dynamics and kinematics—they must become one entity.

For a number of reasons, the mechanics described here is confined to one particle. Simplicity is necessary, especially since much of the discussion is relativistic. The corresponding quantum theory, to use in calculating trajectories, would have to be relativistic as well, and no closed form solutions to relativistic quantum theories are known. For clarity, in explanation, the ideas are given as applied to single-particle motion in some given external fields.

3. HAMILTON–JACOBI FORMALISM APPLIED TO MICROPHYSICS

The entire classically known Hamilton–Jacobi system is not necessary nor even appropriate for the single-particle problem discussed here. Only two quantities from the whole theory are important: (1) the action function $S(x, y, z, t)$, which, once known, can be used to calculate the motion of a particle from any initial position (coordinates are indicated by $x^\mu = (x, y, z, t) = (\mathbf{x}, t)$; (2) the electromagnetic potentials A_μ which are found from Maxwell's equations with given external source currents.

(Both relativistic and nonrelativistic notations are used in this article so that equations will appear in normal form. The Einstein summation convention is in effect. Distance and time are in units of cm, mass in cm^{-1}, hence $c = \hbar = 1$.)

The relativistic equation for S without external forces is

$$\left(\frac{\partial S}{\partial t}\right)^2 - \left(\frac{\partial S}{\partial x}\right)^2 - \left(\frac{\partial S}{\partial y}\right)^2 - \left(\frac{\partial S}{\partial z}\right)^2 = m^2 \tag{2a}$$

or in relativistic notation:

$$\frac{\partial S}{\partial x^{\mu}} \frac{\partial S}{\partial x_{\mu}} = m^2 \tag{2b}$$

And for a particular solution of this equation, Jacobi theory predicts a motion according to

$$\frac{dx}{ds} = \frac{1}{m} \frac{\partial S}{\partial x} \quad \text{or} \quad \frac{dx_{\mu}}{ds} = \frac{1}{m} \frac{\partial S}{\partial x^{\mu}} \tag{3}$$

in which s is an arbitrary path parameter. Equation (1) can be verified by differentiating (3) and using (2) to simplify:

$$\frac{d^2 x_{\mu}}{ds^2} = \frac{1}{m} \left[\frac{\partial}{\partial x_{\rho}} \left(\frac{\partial S}{\partial x^{\mu}} \right) \right] \frac{\partial x_{\rho}}{\partial s} = \frac{1}{2m} \frac{\partial}{\partial x^{\mu}} \left[\frac{\partial S}{\partial x^{\rho}} \frac{\partial S}{\partial x_{\rho}} \right] = \frac{m}{2} \frac{\partial}{\partial x^{\mu}} [1] = 0 \tag{4}$$

The vanishing of the second derivative with respect to the path parameter is the relativistic equivalent of Newton's laws for no acceleration or net force.

Equation (2) reduces to a common nonrelativistic form if S is replaced by

$$S = S' + mt \tag{5}$$

and second-order terms in the derivative with respect to t are dropped:

$$\frac{\partial S}{\partial t} = \frac{1}{2m} \left[\left(\frac{\partial S}{\partial x} \right)^2 + \left(\frac{\partial S}{\partial y} \right)^2 + \left(\frac{\partial S}{\partial z} \right)^2 \right] \tag{6}$$

This equation is useful in many nonrelativistic, low-velocity problems, but the additional mathematical symmetry of the relativistic version (2) is useful, especially for treating the essentially relativistic electromagnetic effects. It is for this reason that both relativistic and nonrelativistic notations are used in this article.

For given initial conditions of velocity and momentum, an infinite number of choices for S are quite possible and physically correct. It is this oversupply of solutions to equation (2) which causes the conceptual complexity of HJ theory and often leads one to solve problems with the Newtonian formula, (1).

A few examples of action functions might be helpful. Suppose $S = Et - px$ then (2) shows that $E^2 - p^2 = m^2$ must hold for particles of mass m and (3) shows that the observed velocity must be p/E. Because (2) is a

first-order linear differential equation, to solve it uniquely, an initial condition must be specified. For instance, a trajectory going through $(t', \mathbf{x}')$ is given by

$$\mathbf{x}=(\mathbf{p}/E)(t-t')+\mathbf{x}' \tag{7}$$

If a different initial velocity is desired, a different action function must be chosen, that is a different $\mathbf{p}$ and E.

Another action which will be used later is

$$S=(t^2-x^2-y^2-z^2)^{1/2}m \tag{8}$$

and has partial derivatives

$$\frac{\partial S}{\partial x}=m\frac{x}{S}, \qquad \frac{\partial S}{\partial t}=-m\frac{t}{S}, \quad \text{or} \quad \frac{\partial S}{\partial x_\mu}=m\frac{x^\mu}{S} \tag{9}$$

For motion effected by this function and for any initial position (x', y', z', t')

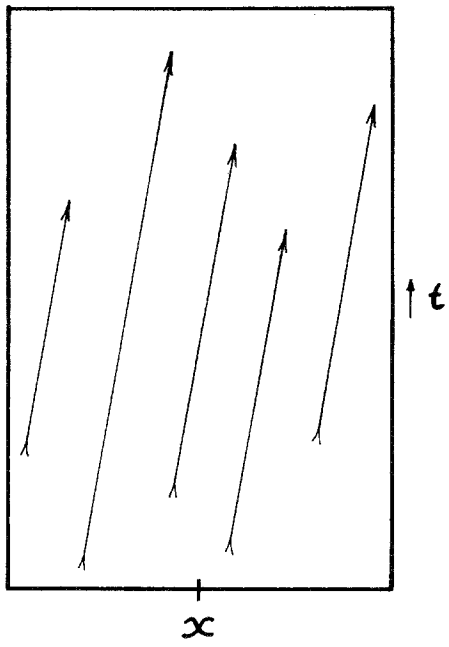

Fig. 1. Trajectories of the two-dimensional action function $S=px-Et$. The energy E and momentum p are considered fixed for the diagram; selection of a trajectory depends on the initial position.

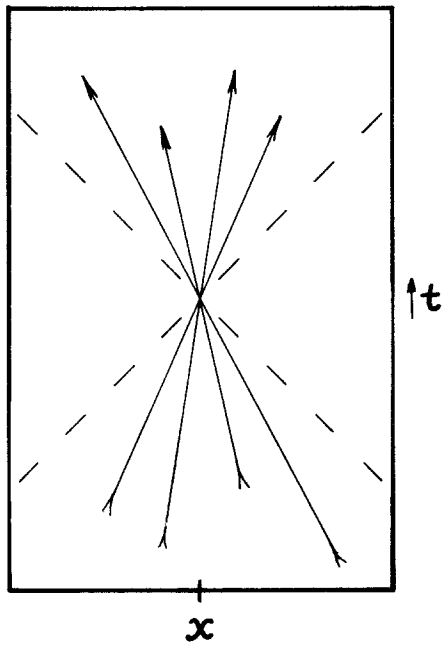

Fig. 2. Trajectories of the function $S=(x^2-t^2)^{1/2}$. This function also represents free-particle motion.

the initial velocity as well as all subsequent velocities must point toward the origin. For this particular S, the initial momentum could alternatively be specified as the initial condition since a position can be found to correspond to any given initial momentum.

A particle which is in motion in this action field is traveling with just the right velocity to pass through the origin at $t=0$. Because (3) is first order, simultaneous initial conditions for both momentum and position cannot be given.

The action functions in these two examples belong to two different well-known classes. The first is a complete integral because the arbitrary parameters E and p can be adjusted to give the motion for any initial velocity and position. A complete integral is really a collection of action functions that is large enough to be able to satisfy any initial condition of position *and* momentum.

The second function has no arbitrary constants; it is a general integral; only the initial position (or initial momentum) is arbitrary. This general integral $S=(t^2-x^2-y^2-z^2)^{1/2}m$ is a single function but describes a collection of trajectories. A complete integral would have to include more

functions. Such could be arranged as

$$S=\left[(t-t')^2-(x-x')^2-(y-y')^2-(z-z')^2\right]^{1/2}m \tag{10}$$

for all (x', y', z', t') on a spacelike surface. The additional parameters are necessary to describe all the solutions of Newton's second-order system using the first-order equation (2). A general integral S contains implied constraints on the possible number and type of trajectory: This property is important to quantum mechanics because, as will be explained, a single wave function is analogous to a single general integral.

Figures 1 and 2 show trajectories for different action functions including several different initial positions. S is physically important because it describes, in a simple way, a class or collection of possible motions, only one of which may actually be used by the particle.

4. THE ELECTROMAGNETIC FIELD IN CLASSICAL PHYSICS

By far the most important force for atomic or microscopic physics is the electromagnetic field. Quantum mechanics without some sort of potential or interaction has essentially no observable consequence. The electromagnetic effects are much better understood than the nuclear or subnuclear interactions, hence it has been chosen as early as 1918, (Weyl, 1950), to form a part of a geometrical theory. The electromagnetic potential, by convention, is symbolized by A_μ and is related to the usual **E** and **B** field via the tensor $F_{\mu\nu}$ by equation (13).

For nonzero electromagnetic field (2) and (3) must have new terms added and thereby become

$$\left(\frac{\partial S}{\partial x_\mu}-eA^\mu\right)\left(\frac{\partial S}{\partial x^\mu}-eA_\mu\right)=m^2 \tag{11}$$

$$m\frac{dx_\mu}{ds}=\frac{\partial S}{\partial x^\mu}-eA_\mu \tag{12}$$

The usual Lorentz force term can be derived by differentiating equation (12) and using (11) analogously to the earlier derivation for force-free motion.

Through the use of the vector potential, the equations become more elegant. The minimal substitution, used in many parts of physics such as quantum electrodynamics and superconductivity, suggests that the vector potential has more significance than a mathematical construct.

For the microscopic discussion at hand, equations (11) and (12) can be simplified. This must be done keeping in mind that the calculational quantities A^μ are defined in terms of the more physical source currents by

$$F_{\mu\nu}=\frac{\partial A_\mu}{\partial x^\nu}-\frac{\partial A_\nu}{\partial x^\mu} \tag{13}$$

and

$$\frac{\partial F^{\mu\nu}}{\partial x^\nu}=4\pi e j^\mu \tag{14}$$

These equations, as well as (11) and (12), permit a gauge transformation

$$A'_\mu=A_\mu+\frac{\partial f}{\partial x^\mu}, \qquad S'=S+\frac{f}{e} \tag{15}$$

whose existence indicates some incompleteness in the usual physical interpretation of the electrodynamic potential. Only the transverse part of A^μ is conventionally considered observable.

In practice, the gauge transformation is important because it is convenient for solving these equations. Equations (13) and (14) can be solved in any simple gauge and the remaining indeterminate part can be regenerated or compensated with the proper change in S. Equations (11) and (12) can then be solved separately using the initial conditions given for the motion. Suppose that subsequent to finding a particular solution of these equations, a gauge transformation with

$$f=-S/e \tag{16}$$

is performed on the equations (11)–(13). The simplification is enormous and (11) and (12) become

$$A_\mu A^\mu=m^2/e^2 \tag{17}$$

and

$$\frac{dx_\mu}{ds}=-\frac{e}{m}A_\mu \tag{18}$$

which may be simplified further by substituting

$$U_\mu=-(e/m)A_\mu \tag{19}$$

to give

$$U_\mu U^\mu = 1 \tag{20}$$

and

$$\frac{dx_\mu}{ds} = U_\mu \tag{21}$$

With these substitutions, equations (13) and (14) become

$$\frac{\partial}{\partial x^\nu}\left(\frac{\partial U^\nu}{\partial x_\nu} - \frac{\partial U^\mu}{\partial x_\nu}\right) = \frac{4\pi e^2}{m} j^\mu \tag{22}$$

In this gauge, the action function is replaced by the remaining degrees of freedom in the electromagnetic potential. These equations are completely equivalent to the earlier case because, in practice, one solves (13) and (14) to get a potential which does not satisfy (17). Following that, a function f is found which, using (15), will satisfy (17). This is equivalent to solving directly the first-order equation (11) for the action function.

This last set of equations (20), (21), and (22), shows a remarkable result which is deeply related to the difference between classical and quantum physics. In simple terms, there are no ultimate constraints on what trajectories of particles are allowed by the equations of motion. A charged particle can always be made to move according to the integral of any vector field, and the source currents to produce those trajectories can be calculated from the given field. Suppose the arbitrary vector field V is given. Define

$$U_\mu = \frac{V_\mu}{\left(V_\mu V^\mu\right)^{1/2}} \tag{23}$$

and choose currents by equation (22). V_μ and U_μ have the same trajectories in 4-space and therefore, also in 3-space. They differ only in the path parametrizations, which are arbitrary.

Classical motion is limited only by the requirements of continuity and differentiability. One would expect to be able to control particles that obey these equations down to the practical limits imposed by the necessity of producing the source current densities j_μ. The limit of the classical theories should be at least as small as the classical radius of the electron, the smallest current element. Surprisingly, from experimental results, the limit is at a much larger size. Quantum mechanical effects do become important and suggest that some modification of equations (11)–(14) is necessary.

Physically, the gauge transformations (15) and (16) mean that the properties of motion usually associated with the action function are electromagnetic in origin. Boundary and initial conditions are always the implied

result of the positions and currents of other charged particles. The initial position and velocity of a particle can be considered as simply the results of all previous electromagnetic interactions.

5. COLLECTIONS OF TRAJECTORIES FOR CLASSICAL MECHANICS

When working with functions S and A_μ as physical quantities, it becomes important to look not at the collection of pairs of functions S and A_μ, which give a certain trajectory, but at the collection of allowed trajectories of a certain fixed S and A_μ. If the external fields and boundary conditions are given functions, it should be possible to find properties of the motion which do not depend on the selection of particular initial conditions. Certainly the selection of a single-particle trajectory can be made by specifying a particular point x^μ on the trajectory, and a collection or ensemble of such trajectories can be specified by a number of such points with each point corresponding to one trajectory in the ensemble. Since, by (17), the trajectory must be timelike, it is appropriate to assume some initial spacelike 3-surface upon which to assign initial conditions.

Any reasonable person would agree that until the position of a particle is known, it could be anywhere. Thus it is, even according to the laws of classical physics. Such a lack of knowledge is expressed by the choice of an initial ensemble distribution which is large and uniform. It is only our intuitive experience which gives us the belief that the initial position of anything could be known. For microscopic single-particle physics, the issue becomes whether the unknown initial position and momentum of a particle could become known infinitely accurately by subsequent physical experimentation. The existence of such a process might explain the intuitive understanding that one has for initial position.

An operational equivalent of "finding a particle" is to bring it to an arbitrary point, such as the origin of the coordinate system, and to hold it there with an action function (such as $S=mt$). This holding trajectory is $x=y=z=0$ for $t>0$.

Suppose there is a way to do this and that somewhere within a very large compact region of volume V there is exactly one charged particle. The particle can be collected and brought to the origin in a simple way. The action function S is selected to be

$$S=\left(x^2+y^2+z^2-t^2\right)^{1/2}m$$

as in equation (8) with t initially sufficiently negative that the entire volume is within the past light cone centered at $t=x=y=z=0$. Solutions of the equations for this function have the required effect. At $t=0$, the particle will

arrive exactly at the origin. The holding function $S=mt$ can then be applied as suggested above, preserving the collected particle. The limit of the collection process comes only through the limits that are imposed by having singular transitions between different action functions.

The discontinuity in S at $t=0$ is important in destroying the possible different incoming momenta. Within the limits of the classical equations of motion the discontinuity can be smoothed so that some small positional uncertainty is left. Even with continuous functions, the volume that contains the particle can be made as small as desired while the attendant momentum spread is zero. The probability of finding the particle in some volume per unit volume is approaching the delta function. In this case, the collapse of the probability density to some small volume is possible for the classical equations of motion. The final position of the particle can be made as independent of the initial position as desired, and consequently the information contained in the initial position can be physically discarded.

This collection procedure provides an operational sequence to set up initial conditions for a particle. The particle can be brought to a known point and later accelerated at will. The importance of this fact is that the classical field equations do not limit the ability to select and move particles.

6. FROM CLASSICAL MECHANICS TO QUANTUM MECHANICS

It is easy to see from the previous discussion that there is no uncertainty principle in classical mechanics as there is in quantum mechanics. It is possible using classical equations to start with a particle at a random position and reduce its uncertainty to a small size, limited only by the practical problems involved. Although, in the classical case, it is not possible to define a wave function, it is possible to form an equivalent density packet. One simply selects elements of the classical ensemble with some distribution that has a packetlike shape. Such a grouping of ensemble elements can be formed physically using methods involving a sequence of action functions to move the particles (or ensemble elements) into place. The dynamics of such a packet follows the classical dynamics of a particle since the ensemble elements make up the packet in a direct way. A fluid model is appropriate. Such a classical wave packet can be formed from an initially uniform ensemble distribution and subsequently made to accelerate, decelerate, expand, and contract arbitrarily.[2] Of course, there are no interference effects, a fact related to the absence of a wave function. Since the

[2]If the packet is contracted to a true delta function, then it cannot be reexpanded. This anomaly cannot be produced in a practical situation and is related to the idealized nature of this function.

particles are conserved, the motion can be described as a fluid of massive noninteracting particles.

All of this is very neat, but it has been observed that the classical laws do not predict the results of experiments correctly. Significant effects occur which cannot be attributed to the size of the electron or other classical limitations. Diffraction and interference, spectral lines, incollapsability of matter, quantum radiation, quantization of angular momentum, and many other manifestations of quantum mechanics indicate that the simple theory presented cannot be right. Contrary to the classical theory, wave packets spread and particles cannot be localized.

Some modification or addition must be made to the basic equations (11)–(15). A term must be added or the interpretation corrected. The existence of nearly singular solutions is suspect since they are not observed, especially in radiation problems. In any case, the form of these equations is not at all the same as the form found commonly in quantum mechanics. There is no wave function as an integrated part of the structure and no exact relationship with modern quantum equations. It is now to be shown that the modification of these equations to quantum mechanics can be carried out with reasonable order and without the loss of the classical foundation or the corresponding trajectories.

Suppose then that some velocity field V_μ exists for the particle even for quantum mechanical situations. Then just as in (3), (12) or (23) there exists a path parameter such that the trajectory can be found by integrating $dx^\mu/ds = V^\mu$. It turns out to be convenient in the quantum mechanical case to let V_μ be unnormalized. There is no physical difference since only the actual 4-space trajectory is observable and the path parameter is not. Basically, to find the trajectories, it is important to decide what element in the field equations must be associated with the velocity. As will be shown later, (12) will suffice as the equation of the particle trajectory if the correct assumption is made for the complex parts of S which develop during the transition to quantum mechanics. From experience with diffraction and interference, $F^{\mu\nu}$ cannot be the simple effective field of force on the particle since it would predict only the usual classical effects.

Some modification of the field equations is required since the change must affect S directly. It turns out that when j^μ is given, (13) can be retained as a definition of $F^{\mu\nu}$; but then, $F^{\mu\nu}$ has the usual experimental meaning only in the classical limit.

The most important changes are to equation (11). A simple modification can be made by adding a small second-order term. One of the simplest such second-order gauge-invariant quantities is

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial S}{\partial x_\mu} - eA^\mu\right)$$

Let

$$p^\mu = \frac{\partial S}{\partial x_\mu} - eA^\mu \tag{24}$$

be the cannonical momentum so that (11) becomes

$$p^\mu p_\mu = m^2 \tag{25}$$

then the new term would make this equation

$$p_\mu p^\mu + a\frac{\partial}{\partial x^\mu} p^\mu = m^2 \tag{26}$$

in which a is a small constant. a may be estimated by studying when the second term will become important with respect to the first term. Arguing qualitatively, if p makes a significant fractional change, $dp/p \sim 1$, over a distance Δx then the second term will be approximately the same size as the first if $\Delta p \sim p \sim a/\Delta x$ or if $\Delta p \Delta x \sim a$. This suggests a change in the character of the solutions when $\Delta p \Delta x$ becomes too small.

With foresight of the uncertainty principle, it is easy to suggest that the magnitude of a be approximately that of Planck's constant $\hbar$. In fact, $a = i\hbar$ gives a system of equations that is equivalent to quantum mechanics. To see this explicitly, the substitutions $S = -i\hbar \ln \psi$ and $a = i\hbar$ in equation (26) gives

$$\left(\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - eA_\mu\right)\left(\frac{\hbar}{i}\frac{\partial}{\partial x_\mu} - eA^\mu\right)\psi = m^2\psi \tag{27}$$

If the quantities a and S are interpreted correctly, the physical properties of the modified equation (26) are equivalent to the physical properties of equation (27), whose correctness and validity is long established. This algebraic transformation is little known and is hidden by the linear form of (27). Linearity, a long-emphasized, essential property of quantum mechanics, is important for the practical solution of even the simplest problems; but, nevertheless, equation (26) is physically simpler.

With this substitution, the function S, associated with the action function of classical mechanics, becomes complex. S can no longer be considered completely classical. This transition was treated by Bohm (1951) by using $Re^{iS} = \psi$ and keeping S and R real. In any case, for a situation in which the motion can be described classically, the S used here can always be chosen real while Bohm's R can always be set equal to 1. If the motion is quantum, then R is nonconstant or, equivalently, $\mathrm{Im}(S)$ is nonconstant. The single complex variable S, which is the form originally used by this author,

has been found more convenient in the geometrical discussions. Especially in the following article, the close physical relation of S, ψ, A_μ and the metric tensor allows a smaller number of labeled fields.

This interpretation of the uncertainty principle depends both upon the concept of trajectory and this form of the field equation. It is the nonlinear term in (26) which causes a deflection in the trajectory. This constraint can be viewed as a constraint on the action function S or wave function $\psi = e^{iS}$ which limits the correlation between the possible velocities of the particle and a range of possible initial conditions. In this context, the uncertainty principle is due to the relative constraint put upon x_μ and p_μ by the use of a single constrained function to describe the motion.

Since the quantum mechanical equivalent of the classical cannonical momentum is gauge invariant, the gauge transformation (15) is still allowed for the electromagnetic potential, but the action function S must also be changed during the transformation as for the classical case. For both the classical equation (25) and the quantum equations (26) and (27), it is possible to eliminate the action function (or wave function) by performing a gauge transformation. As discussed earlier, for the classical system, the result is (17), (19), and (22). For the quantum system the corresponding field equation is

$$e^2 A_\mu A^\mu + ei\hbar \frac{\partial}{\partial x_\mu} A_\mu = m^2 \tag{28}$$

and in both cases, the electromagnetic potential would be restrained by the additional equation either (14) or (22).

7. TRAJECTORIES FOR QUANTUM MECHANICAL MOTION

Several important physical questions relate to this form of the quantum equations. It is surprising, at first, that a is so small; but, in reality, it is just that people are large for need of complexity and are made of small structural elements. The use of a complex parameter is contrastingly harder to reconcile. In this gauge, because of (13) and (14), A_μ must contain a nonzero real part when there are any external source currents. Equation (28) then implies that insofar as there are accelerations, A_μ must contain a nonzero imaginary part. This imaginary part, in the context of this paper, defies sensible physical interpretation especially with regard to the possibility of an observable velocity vector field. Further interpretation is included in the succeeding article. The real part of A_μ is the closest analog to the classical A_μ while, from the mathematics, the complex numbers are forced by the need for a linear equation having a simple relationship to the

probability density. The classical A_μ given by the system (13), (14), and (17) can be interpreted as a velocity by (19) because of the simple form of (17) and the fact that A_μ can always be selected real.

In both of these systems of equations the question of uniqueness is important, and in most practical situations these equations have unique solutions given the appropriate boundary or initial conditions. An ambiguity in the solution usually indicates, as in the rest of physics, that there is an unanswered physical question. Details of uniqueness criteria are outside the scope of this paper.

The character of equation (28) near a singularity is strongly controlled by the second term. The solutions used in the previous section to gather particles, or more properly, ensemble elements, are no longer allowed because of the large singularity at the origin at $t=0$. This singularity is necessary to stop particles as they arrive at the origin in a way that is independent of their incoming direction. The constraint on the possible motions of the particles is contained in the field equation for A_μ and does not bear on the question of the existence of trajectories derived from the vector potential field. This opens the possibility of a classical type of hidden variable theory in which the trajectories can be defined in a meaningful mathematical way but the particles cannot be localized or predicted because of the necessity of directing them with the severe constraining limit of the nonlinear equation (28) (Madelung, 1926).

The collection process of the previous section is related to the properties of spreading wave packets. There are no known closed form wave packet solutions to (28), and, for this reason, an understandable, physically straightforward discussion is easiest in the nonrelativistic limit. (See the related interpretation, Lamb, 1969.)

To derive a nonrelativistic wave packet, begin again with the relativistic action function (8) and assume as before that a sufficient amount of time is allowed for the particles to come to the origin without the need for relativistic velocities. In this case, $\Delta x/\Delta t \ll 1$, at least away from the origin, and a nonrelativistic approximation may be used:

$$S=mt+mt\left(1-\frac{x^2+y^2+z^2}{t^2}\right)^{1/2}$$

$$\approx mt-\frac{m}{2}\frac{r^2}{t} \tag{29}$$

This approximation does not hold near $t=0$, and in this range, another form for S must be used.

This action function can be easily compared to a physically similar solution of the Schrödinger equation. The exact quantum wave packet

solution can be found by taking the nonrelativistic limit of the relativistic equation (27), and then solving it for the appropriate boundary conditions. In all nonrelativistic limits, it is convenient to redefine the action in accordance with equation (5) so as to exclude the linear time dependence term in (29). A corresponding change, to a new wave function is also necessary:

$$\psi' = \psi e^{imt} \tag{30}$$

Assuming also that $V \ll m$ and $(d/dt)(\ln V) \ll mc/\hbar$ and that the resulting time dependence of the wave function ψ' is slow enough that the second-order time derivative is negligible, an approximation to (27) gives the nonrelativistic Schrödinger equation in Cartesian coordinates:

$$\frac{1}{2m} \sum_{k=1}^{3} \left(\frac{\hbar}{i} \frac{\partial}{\partial x_k} - eA^k \right)^2 \psi' = \left(-i \frac{\partial}{\partial t} + V \right) \psi' \tag{31}$$

The free-particle solution of this equation corresponding to an optimally converging wave packet is well known:

$$\psi' = \exp\left[-\frac{r^2/4}{(\Delta r_0)^2 + it/2m} \right] \cdot (2\pi)^{-3/4} \cdot \left[(\Delta r_0)^2 + \frac{1}{2} \frac{it}{m \Delta r_0} \right] \tag{32}$$

This solution differs from the earlier classical solution (29) by the displacement of the singularity off of the real time axis by the small amount $2m \cdot (\Delta r_0)^2$ and by the slowly varying normalization term. The normalization factor makes the wave function the carrier of probability information as well as of the motion and is required by the differential equation. Note that even though there is no minimum width of the packet, the quantum mechanical problem of stopping the particle at $t = 0$ is not solvable because the solution cannot be continued to the same solution $S = mt$ as would be possible for the classical field equation. The Schrödinger equation is not algebraic in the derivatives of S, and the higher-order singularities are not allowed to the extent that $\hbar$ is nonzero. In the right sort of experiment, the second-order term becomes important before any effects of coarseness in the source currents can influence the result. Because the Schrödinger equation for the wave function is essentially a differential equation, a first-order discontinuity is not allowed, and the solutions which would otherwise permanently reduce the width of the statistical distribution of position are also not allowed. This argument is related to the quantum mechanical fact that the matching of the derivatives of the wave function at the boundary is necessary for the conservation of probability current. The wave function has

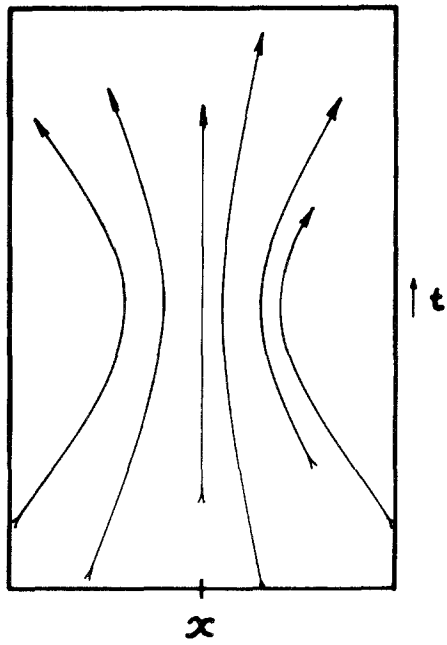

Fig. 3. Trajectories of particles in the x–t plane moving according to the wave function of a converging wave packet. This diagram is the quantum equivalent of Figure 2.

a certain amount of extra rigidity, imparted by the differential equation. The calculated trajectories for the wave packet solution are drawn in Figure 3.

Having introduced these concepts, it is an easy matter to define the trajectories for an arbitrary wave function. It can be shown that these trajectories, if each is assigned the correct probability, give a probability density variation which is in accord with quantum mechanics. As mentioned, the vector potential may become complex. The imaginary part represents some degree of freedom intrinsic to quantum mechanics and of open interpretation. If the particle is allowed to exist off of the real axis, the interpretation of probability density becomes difficult.

The extra dimensionality that would be generated does not have a sensible physical interpretation, and consequently, the proper procedure must be to take the real part of the velocity that would be given by simply substituting the complex wave function directly:

$$U_\mu = \mathrm{Re}\left(\frac{\partial S}{\partial x^\mu} - eA_\mu\right) \tag{33}$$

The conserved current density J^μ for the particles moving in the wave function can be found if the correct multiplier for the velocity can be found. This multiplier corresponds roughly to the reference density or calibration that must be made to measure the dependence of the probability density in space and time. This reference density corresponds physically to the existence of a pure state. A single isolated particle is always coherent with itself; it has a single well-defined wave function, and therefore, by virtue of the statement that that wave function can be produced, it must have a single virtual source which gives it the proper single reference density. For an undiffracted free particle, this reference density is uniform over all space-time.

Only one multiplier function is allowed for a given wave function up to a normalization constant. The current density must be of the form

$$J_\mu = f(x)U_\mu(x) \tag{34}$$

for some function $f(x)$. The correct function $f(x)$ is given by the condition for current conservation

$$\frac{\partial}{\partial x^\mu} J^\mu = 0 \tag{35}$$

which implies that

$$\frac{\partial}{\partial x^\mu}\left[f(x)U^\mu(x)\right] = 0 \tag{36}$$

The known probability density current for the relativistic equation (27) which satisfies (35) is usually given in the gauge having A real:

$$\begin{aligned} J_\mu &= \frac{1}{2}\left[\psi^*\left(\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} - eA_\mu\right)\psi - \psi\left(\frac{\hbar}{i}\frac{\partial}{\partial x^\mu} + eA_\mu\right)\psi^*\right] \\ &\equiv \psi\psi^* \mathrm{Re}\left(\frac{1}{i}\frac{\partial}{\partial x^\mu}\ln\psi - eA_\mu\right) \end{aligned} \tag{37}$$

This shows that up to a normalization factor, the correct choice for $f(x)$ is $f(x) = \psi\psi^*$.

The conservation of probability current can be proved from the field equation using integration by parts in the conventional way. This integration, in particular, fails if all coordinates are not to remain real.

In the same nonrelativistic limit as is appropriate for the nonrelativistic field equation, the current density becomes, with $S' = S - mt$,

$$J_0 = \psi\psi^*$$

$$J_k = \frac{\hbar}{2m}\left[\psi^*\left(\frac{\partial}{\partial x_k} - \frac{e}{\hbar}A_k\right)\psi - \psi\left(\frac{\partial}{\partial x_k} + \frac{e}{\hbar}A_k\right)\psi^*\right]$$

These currents are well known for quantum mechanics and have been discussed before in the literature.

8. THE HIDDEN VARIABLES OF BOHM

An earlier theory by Bohm (1951, 1952) is similar to the trajectory approach explained here. Many parts of those papers are instructive for persons interested in the possible relationship of these trajectories to the usual interpretation of quantum theory. Bohm's theory is nonrelativistic and contains a simplified form of the electromagnetic interaction. The electromagnetic potential is not used as an explicit source of the velocity, and the various field functions are not integrated by gauge transformations. The overwhelming interrelation of quantum effects and the electromagnetic field is not in perspective.

A wave function is not a simple thing. It must be representative of all interference effects from all interactions that ever occur to the particle. To think of a particle as always traveling in a simple well-formed wave packet is extremely naive. The physical association of a naive wave packet with a point particle leads to some difficulties when attempting to geometrize the quantum equation. This tight association of wave and particle prohibits physical integration of the wave function with the electromagnetic fields. Consequently, the structure becomes irreconcilably at odds with the fundamental concept of geodesic.

Once the wave function and the electromagnetic potential are gauged together, the uncertainty principle becomes a property of the field equations and is irrelevant to the existence of trajectories. The physical equivalence of the wave function to electromagnetic fields suggests that a wave function free set of equations for trajectories might be possible. See the nonquantum version by Feynman and Wheeler (1945). The effect of the quantum field equation makes it impossible to set up (or measure) a point particle with complete position and momentum information. Furthermore, the physics is complicated by the classically well-understood process whereby, during a two-particle interaction, the resultant momentum and position of either one of the particles depends on the initial momentum and position of both particles. The information in the variables is mixed up by the collision.

Suppose that one is trying to discover position or momentum coordinate information about a test particle by use of probe particles. The momentum and position of the probe particle can only be half known; and after collision, because of the exchange of coordinate dependency, the position and momentum of the test particle will be less well known. This is the process of randomization by collision used by Bohm. The argument that such interactions of the test particle leads to randomization seems correct if the unknown and a priori unknowable hidden variables of the randomizing

probe particles really can be assumed random and not just unknown. If one is to discover some pattern of nonrandomness in the absolute initial coordinates of some finite collection of probe particles, it would be necessary to distinguish those initial coordinates from the effects of all interposing interactions. Such a process is impossible for the electromagnetic particles studied here, because any finite collection of particles has interacted overwhelmingly with other particles during its existence.

To measure a probability density and test the $P=\psi\psi^*$ law requires an ensemble of experiments. Each experiment creates a wave function that is similar to a particular standard wave function characteristic of that experiment. This process of wave function preparation is essential to the operational definition of the ensemble and therefore the probability density. For each particle, each step of the preparation involves an interaction with at least one additional particle which at best may be treated classically. A specific example is the attempted creation of the state $\psi=e^{imt}$ in the previous section.

A classically random ensemble is random for quantum motion as well. No decrease in randomness of the "hidden variables" is possible when further constraining these particles to quantum motion. Because of the complex way information is exchanged in a collision and because of the large amount of presumed random initial data, it is not possible to exceed the uncertainty limits that are intrinsic to the quantum field equations.

9. CONCLUSION

This article is an attempt to conceptually resynthesize some of the basic ideas of physics that have been at issue since the discovery of quantum mechanics. Many of these misconceptions have, in the opinion of the author, impeded progress in the deep understanding of fundamental physical processes. The conceptualization presented is simple but different from the many others.

The wave function is a function of utility, used to calculate the motion of particles and the average properties of that motion. The wave function is physically and mathematically merged with the electromagnetic potentials. The differences between the results of quantum and classical calculations are due entirely to the addition of a single small term, proportional to the field equation. When described in the proper context, very little other modification is required to explain the differences between classical equations and quantum equations of motion.

ACKNOWLEDGMENTS

I would like to thank Professor E. Wichtmann for some early discussions about particle trajectories and Professor Sumner Davis for helpful comments on the final manuscript. Some

of the preliminary concepts for this paper were developed while the author held an N.S.F. fellowship.

REFERENCES

Bohm, D. (1951). *Physical Review*, **85**, (2), 166–179.
Bohm, D. (1951). *Physical Review*, **85**, (2), 180–193.
Bohm, D. (1952). *Physical Review*, **89**, (2), 458–466.
Feynman, R. P., and Wheeler, A. (1945). *Reviews of Modern Physics*, **17**, (2), 157–181.
Galehouse, D. C. (1981). Geometrical Derivation of the Klein–Gordon Equation, *International Journal of Theoretical Physics*, **20**, (6), 457.
Lamb, W. (1969). *Physics Today*, **22**, (4), 23–28.
Madelung, E. (1926). *Zeitscrift für Physik*, **40**, 322–326.
Newton, I. (1934). *Principia*, Vol. I, University of California Press.
Weyl H. (1950). In H. A. Lorentz, *The Principle of Relativity*. Dover, New York.